

THE RECURRENCE TIME FOR ERGODIC SYSTEMS OF INFINITE MEASURES

STEFANO GALATOLO, DONG HAN KIM, AND KYEWON KOH PARK

ABSTRACT. We investigate quantitative recurrence in systems having an infinite measure. We extend the Ornstein-Weiss theorem for a general class of infinite systems estimating return time in decreasing sequences of cylinders. Then we restrict to a class of one dimensional maps with indifferent fixed points and calculate quantitative recurrence in sequences of balls, obtaining that this is related to the behavior of the map near the fixed points.

1. INTRODUCTION

The well known Poincaré recurrence theorem states that under suitable assumptions a typical trajectory of the system comes back infinitely many times in any neighborhood of its starting point. This result does not give an estimation about the time necessary for an orbit to come back near the starting point. This naturally raises questions like: how much iterations of an orbit is necessary to come back within a distance r from the starting point? The *quantitative recurrence* theory investigates this kind of questions.

In the literature such indicators of quantitative recurrence have been defined in several ways by measuring the rate of the first return time of an orbit with respect to a decreasing sequence of neighborhoods of the starting point. A natural approach is to use the metric of the space X and consider a decreasing sequence of balls ([7], [4], [3], [25], [26], [18], [16]) in the above question. More precisely, let us define $\tau_r(x)$ to be the first return time of x into the ball $B(x, r)$ centered in x and with radius r . It turns out that in many chaotic systems the asymptotic behaviour of $\tau_r(x)$, when r is small is related to the local dimension of the invariant measure at x :

$$(1.1) \quad \tau_r(x) \simeq r^{d_\mu(x)}.$$

This is proved for a.e. x in “fast” mixing systems ([25]). And in general systems the asymptotic behavior of $\tau_r(x)$ only gives a lower bound for the dimension.

Other interesting sequences of neighborhoods of x can also be defined, using the symbolic dynamics generated by a partition ξ . Let $\xi_k = \vee_{i=0}^{k-1} T^{-i}\xi$. We consider a decreasing sequence of cylinders $\xi_k(x)$ containing x . Here we define $R_k(x)$ to be the smallest n such that $T^n(x) \in \xi_k(x)$. A partition $\xi = \{\xi^0, \dots, \xi^{\ell-1}\}$ naturally associates to a symbolic name to a.e. x . The symbolic name $(x_i)_{i=1}^\infty$ of x is given by the rule

$$x_i = k \iff T^i x \in \xi^k, \quad k = 0, \dots, \ell - 1.$$

Hence $R_k(x)$ denote the first time greater than k that is necessary to see the k -name of x again in the orbit of x . It is well known that the asymptotic behavior of $R_k(x)$

The third author is supported in part by KRF R05-2004-000-10864-0.

as k increases is related to the entropy of the system. If ξ is generating then for a.e. x ([23],[24])

$$R_k(x) \simeq 2^{h_\mu(T)k}.$$

Other quantitative recurrence indicators consider the forward images of the whole cylinder ξ_k and consider the first return for the cylinder, which is the minimum n satisfying $T^n(\xi_k) \cap \xi_k \neq \emptyset$ ([26],[6]).

In computer simulations or experimental situations, the quantitative recurrence indicators can be easily estimated by looking at the behavior of a “typical, random” orbit and its first return time in a sequence of sets centered around the starting point. By the above relations recurrence and related concepts were used to estimate the dimension of attractors (see [15] and references therein, or [8], [13], [10], [9], [16]) or entropy (see e.g. [12]).

The aim of this work is to investigate the quantitative recurrence in dynamical systems preserving an infinite measure. Such systems are used for models of statistically anomalous phenomena such as intermittency and anomalous diffusion (e.g. [22][14]) and they do have interesting statistical behavior. Many classical theorems of finite measure preserving systems from ergodic theory can be extended to the infinite measure preserving case. In particular we cite the Hopf ratio ergodic theorem which will be used later. Let T be conservative and ergodic (see. e.g. [2]) and $f, g \in L^1$ such that $\int f d\mu, \int g d\mu \neq 0$, then

$$\frac{\sum_{k \in [0, n-1]} f(T^k(x))}{\sum_{k \in [0, n-1]} g(T^k(x))} \rightarrow \frac{\int f d\mu}{\int g d\mu}$$

as $n \rightarrow \infty$ for a.e. x .

The notion of entropy can be extended to the infinite case (see e.g. [28], [21]). Let T be a conservative, ergodic measure preserving transformation on a σ -finite space (X, \mathcal{A}, μ) . Then the entropy of T can be defined as

$$h_\mu(T) = \mu(Y) h_{\mu_Y}(T_Y)$$

where $Y \in \mathcal{A}$ with $0 < \mu(Y) < \infty$ and T_Y is the first return map of Y ($T_Y(x) = T^{R_Y(x)}(x)$ where

$$R_Y(x) = \min\{n \geq 1 : T^n(x) \in Y\}$$

when $x \in Y$) and μ_Y is the induced measure ($\mu_Y(E) = \frac{\mu(E \cap Y)}{\mu(Y)}$) which is invariant and ergodic under T_Y . Since this notion does not depend on the choice of Y , we can well define the entropy of infinite measure preserving systems as above.

To give examples of the particular behavior of infinite measure preserving maps let us consider a particularly representative class of maps on the unit interval: the Manneville-Pomeau maps defined by:

$$(1.2) \quad T(x) = \begin{cases} x + 2^{z-1}x^z & \text{if } 0 \leq x < 1/2, \\ 2x - 1 & \text{if } 1/2 \leq x < 1. \end{cases}$$

these maps have an indifferent “slowly repulsive” fixed point at the origin. When $z \in [2, \infty)$ this forces the natural invariant measure for this map to be infinite and absolutely continuous with respect to Lebesgue. It is not hard to see that $\xi = \{[0, 1/2), [1/2, 1)\}$ is a generating partition and the entropy $h_\mu(T)$ is positive and finite. Moreover we have a version of the Shannon-McMillan-Breiman Theorem

([17]): for every $f \in L^1(\mu)$, with $\int f \neq 0$

$$\frac{-\log(\mu(\xi_n(x)))}{S_n(f, x)} \rightarrow \frac{h_\mu(T)}{\int f d\mu}$$

as $n \rightarrow \infty$ for a.e. x . Here $\xi_n(x)$ is the element of the iterated partition $\xi_n = \bigvee_{i=1}^{n-1} T^{-i}\xi$ containing x and $S_n(f, x)$ is the partial sums of f along the orbit of x :

$$S_n(f, x) = \sum_{k \in [0, n-1]} f(T^k(x)).$$

This gives among other applications a result showing how the behavior of these infinite systems is “weakly chaotic” ([28], [5]) giving symbolic orbits which have an algorithmic information content growing as a sublinear power law as time increases, while for a system having finite measure and positive entropy the information content increases linearly with time.

In this work we generalize Ornstein-Weiss theorem to a wide class of conservative, ergodic, infinite measure preserving systems. More precisely we prove that :

$$R_n(x) \simeq 2^{\frac{h_\mu(T)}{\alpha \int f d\mu} S_n(f, x)}$$

almost everywhere, where α is a constant depending on the system under consideration and f and S_n are as above. This is proved in a weak form (Theorem 2.3) with a “lim sup” statement for general systems. The statement is also proved in a stronger form (Theorem 3.2) for a system satisfying a strong mixing condition.

In section 4 we apply the previous result to a class of smooth maps on the interval with a neutral fixed point containing the Manneville-Pomeau family (1.2) and we calculate for this family the quantitative recurrence indicators into balls, showing that they are related to the behavior of the map near to the neutral fixed point and not on the local dimension (recall (1.1)). More precisely, we see that if a map is of the form as in (1.2) then

$$\tau_r(x) \simeq r^{z-1}$$

as $r \rightarrow 0$ (see theorem 4.1 for the general statement).

In the last section we show an example of a system having a particular behavior of recurrence. In this system the asymptotic behavior of $R_n(x)$ oscillates so that $\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n}$ does not exists.

Acknowledgments. The authors wish to thank J-R Chazottes for fruitful discussions, Korea Institute for Advanced Study and Scuola Normale Superiore for logistic help and for the nice environment where the work has been done.

2. GENERAL CASE

Let us consider a measure preserving system (X, T, \mathcal{A}, μ) . Let $S_n(A, x)$ be the number of $T^i x \in A$ for $0 \leq i \leq n-1$, i.e.,

$$S_n(A, x) = S_n(1_A, x) = \sum_{i=0}^{n-1} 1_A(T^i(x)).$$

Let ξ be a measurable partition of X and let

$$\xi_n = \xi \vee T^{-1}\xi \vee \dots \vee T^{-n+1}\xi.$$

Given a partition ξ we denote the atom containing x by $\xi(x)$. Define $R_n(x)$ by

$$R_n(x) = \min\{j \geq 1 \mid \xi_n(x) = \xi_n(T^j x)\}$$

considering a fixed set $A \in \mathcal{A}$ we also define $\bar{R}_n(x)$ by

$$\bar{R}_n(x) = \min\{S_j(A, x) \geq 1 \mid \xi_n(x) = \xi_n(T^j x)\}.$$

Note that $\bar{R}_n(x) = S_{R_n(x)}(A, x)$.

Lemma 2.1. *Let T be a conservative, ergodic measure preserving transformation (c.e.m.p.t.) on the σ -finite space (X, \mathcal{A}, μ) with $h_\mu(T) < \infty$, and let ξ be a finite generating partition (mod μ). Assume that there is an atom A in ξ with $0 < \mu(A) < \infty$. For almost every $x \in A$*

$$\lim_{n \rightarrow \infty} \frac{\log \bar{R}_n(x)}{S_n(x)} = \frac{h_\mu(T)}{\mu(A)}.$$

Proof. Let ξ_A be the induced partition on A , where

$$\xi_A = \cup_{k \geq 1} \{V \cap \{R_A = k\} : V \in A \cap \xi_k\}.$$

Since a generating partition induces a generating partition for the induced map, the induced map T_A on A is isomorphic to the shift map on $(\xi_A)^\mathbb{N}$. Let $\tilde{x} = (\tilde{x}_i)_{i=1}^\infty \in (\xi_A)^\mathbb{N}$, $\tilde{x}_i \in \xi_A$ be the corresponding point of x in $(\xi_A)^\mathbb{N}$. Then

$$\begin{aligned} R_n(\tilde{x}) &= \min\{j \geq 1 \mid \tilde{x}_1 \dots \tilde{x}_n = \tilde{x}_{j+1} \dots \tilde{x}_{j+n}\} \\ &= \min\{j \geq 1 \mid (\xi_A)_n(x) = (\xi_A)_n(T_A^j x)\}. \end{aligned}$$

By the Ornstein-Weiss theorem for almost every x we have

$$\lim_{n \rightarrow \infty} \frac{\log R_n(\tilde{x})}{n} = h_{\mu_A}(T_A, \xi_Y) = h_{\mu_A}(T_A).$$

Let $m(x, n)$ be a positive integer defined by

$$m = R_A(x) + R_A(T_A x) + \dots + R_A(T_A^{n-1} x).$$

Then we have $(\xi_A)_n(x) = \xi_m(x)$. Since $T_A^j = T^{S_j(x)}(x)$, we have

$$\begin{aligned} R_n(\tilde{x}) &= \min\{j \geq 1 \mid (\xi_A)_n(x) = (\xi_A)_n(T_A^j x)\} \\ &= \min\{S_j(x) \geq 1 \mid \xi_m(x) = \xi_m(T^j x)\} = \bar{R}_m(x). \end{aligned}$$

If $S_k(x) = n$ then $m(x, n-1) < k \leq m(x, n)$. Thus we have

$$R_{n-1}(\tilde{x}) = \bar{R}_{m(n-1)}(x) \leq \bar{R}_k(x) \leq \bar{R}_{m(n)}(x) = R_n(\tilde{x})$$

and

$$\frac{\log R_{n-1}(\tilde{x})}{n} \leq \frac{\log \bar{R}_k(x)}{S_k(x)} \leq \frac{\log R_n(\tilde{x})}{n}.$$

Hence for almost every $x \in A$ we have

$$\lim_{k \rightarrow \infty} \frac{\log \bar{R}_k(x)}{S_k(x)} = \frac{h_\mu(T)}{\mu(A)}.$$

□

Let us recall the following fact also known as the Aaronson ergodic theorem ([2]).

Theorem 2.2. *Suppose that T be a c.e.m.p.t. of the σ -finite, measure space (X, \mathcal{B}, m) , and let*

$$a(n) \uparrow \infty, \frac{a(n)}{n} \downarrow 0 \text{ as } n \uparrow \infty,$$

then (1) If $\exists A \in \mathcal{B}$, $0 < m(A) < \infty$ such that

$$\int_A a(R_A) dm < \infty,$$

then

$$\frac{S_n(f)}{a(n)} \rightarrow \infty \text{ a.e. } \forall f \in L^1(m)_+$$

(2) Otherwise,

$$\liminf_{n \rightarrow \infty} \frac{S_n(f)}{a(n)} = 0 \text{ a.e. } \forall f \in L^1(m)_+$$

Theorem 2.3. Let T be a c.e.m.p.t. on the σ -finite space (X, \mathcal{A}, μ) with $h_\mu(T) < \infty$, and let $\xi \subset \mathcal{A}$ be a finite generating partition (mod μ). Assume that there is an atom A in ξ with $0 < \mu(A) < \infty$. Then for any $f \in L^1(\mu)$ with $\int f d\mu \neq 0$,

$$\limsup_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(f)(x)} = \frac{h_\mu(T)}{\alpha \int f d\mu} \text{ a.e.,}$$

almost everywhere, where

$$\alpha = \sup\{\beta : \int_A (R_A)^\beta dm < \infty\}.$$

Moreover, if $\alpha = 0$, then the limsup goes to infinity.

Proof. For any C if n is large, then by Theorem 2.2 (1)

$$S_n(x) \geq Cn^{\alpha-\epsilon},$$

so for large k

$$\bar{R}_k(x) = S_{R_k(x)}(x) \geq C(R_k(x))^{\alpha-\epsilon}.$$

Taking the logarithm we have

$$\log \bar{R}_k(x) \geq (\alpha - \epsilon) \log R_k(x) + \log C.$$

which implies

$$\frac{\log R_k(x)}{\log \bar{R}_k(x)} \leq \frac{1}{\alpha - \epsilon} \left(1 - \frac{\log C}{\log \bar{R}_k(x)} \right).$$

Hence we conclude

$$(2.1) \quad \limsup_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} \leq \frac{1}{\alpha}.$$

For the reversed inequality, suppose that there is $\eta > 1$ such that

$$\limsup_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} \leq \frac{1}{\eta^3 \alpha}.$$

Then for all k large enough we have $(R_k(x))^{\eta^2 \alpha} < \bar{R}_k(x)$. Since $\bar{R}_k(x) = S_{R_k(x)}(x)$, if $S_n = \bar{R}_k$ and $n = R_k$ for k large enough, we have

$$(2.2) \quad S_n > n^{\eta^2 \alpha}.$$

But by Theorem 2.2 and the definition of α we have infinitely many n 's such that

$$S_n(x) < n^{\eta \alpha}.$$

Let n_0 be such an n and $m = S_{n_0}(x) < n_0^{\eta \alpha}$. Then, since S_n is increasing, if $m < S_\ell < m^\eta$, then $\ell > n_0$ and

$$S_\ell < m^\eta < n_0^{\eta^2 \alpha} < \ell^{\eta^2 \alpha},$$

which implies that $S_\ell \neq \bar{R}_k$ for any k by (2.2). Therefore, there are infinitely many m 's such that

$$(2.3) \quad \bar{R}_k \notin (m, m^n) \text{ for any } k.$$

By Lemma 2.1 for any $\epsilon > 0$, if k is large, then

$$\left| \frac{\log \bar{R}_k(x)}{S_k(x)} - \frac{h_\mu}{\mu(A)} \right| < \epsilon.$$

Since S_k increases by 1 at most, if k is large, we have

$$(2.4) \quad \bar{R}_{k+1} < 2^{S_{k+1}(\bar{h}+\epsilon)} \leq 2^{(S_k+1)(\bar{h}+\epsilon)} = 2^{\bar{h}+\epsilon} (2^{S_k(\bar{h}-\epsilon)})^{\frac{\bar{h}+\epsilon}{\bar{h}-\epsilon}} < 2^{\bar{h}+\epsilon} (\bar{R}_k)^{\frac{\bar{h}+\epsilon}{\bar{h}-\epsilon}},$$

where $\bar{h} = h_\mu/\mu(A)$. Therefore, for large k we have $\bar{R}_{k+1} < \bar{R}_k^\eta$ which contradicts to (2.3). Hence we have

$$\limsup_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} \geq \frac{1}{\alpha}$$

and combined with (2.1) we have

$$\limsup_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} = \frac{1}{\alpha}.$$

By applying Lemma 2.1 we obtain

$$\limsup_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(x)} = \frac{h_\mu(T)}{\alpha \mu(A)}.$$

The result follows replacing 1_A by Hopf's ergodic theorem with any $f \in L^1$ with $0 < \int f d\mu < \infty$. \square

3. MIXING CASE

A set A is called a Darling-Kac set, if there is a sequence $\{a_n\}$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n \hat{T}^k 1_A = \mu(A)$$

almost uniformly on A . A function f is slowly varying at ∞ if

$$\frac{f(xy)}{f(x)} \rightarrow 1 \text{ as } x \rightarrow \infty, \forall y > 0.$$

Suppose that T has a Darling-Kac set, whose return time process is uniformly - or strongly mixing from below (see [1] e.g.) and

$$(3.1) \quad a_n(T) \sim n^\alpha s(n),$$

where $s(n)$ is a slowly varying function. The Darling-Kac Theorem states that

$$\frac{S_n(x)}{a_n(T)} \rightarrow Y_\alpha, \text{ in distribution,}$$

where Y_α has the normalized Mittag-Leffler distribution of order α . By Theorem 2.2, α in (3.1) is the same α in Theorem 2.3.

By Aaronson's work we have the following theorem:

Fact 3.1 ([1]). Let $L_2(n) = \log(\log n)$ and $K_\alpha = \frac{\Gamma(1+\alpha)}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$.

$$\limsup_{n \rightarrow \infty} \frac{1}{n^\alpha s(n/L_2(n)) L_2(n)^{1-\alpha}} \sum_{k=1}^n f \circ T^k = K_\alpha \int f d\mu$$

a.e. for every $f \in L_+^1$.

Thus, for any $\epsilon > 0$ we have for almost every x

$$(3.2) \quad \limsup_n \frac{S_n(x)}{n^{\alpha+\epsilon}} = 0.$$

Theorem 3.2. Let T be a c.e.m.p.t. on the σ -finite space (X, \mathcal{A}, μ) with $h_\mu(T) < \infty$, and let $\xi \subset \mathcal{A}$ be a finite generating partition (mod μ). Assume that there is an atom A in ξ with $0 < \mu(A) < \infty$. Suppose that T has a Darling-Kac set, whose return time process is uniformly - or strongly mixing from below and

$$a_n(T) \sim n^\alpha s(n),$$

where $s(n)$ is a slowly varying function. Then for any $f \in L^1(\mu)$ with $\int f d\mu \neq 0$,

$$\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(f)(x)} = \frac{h_\mu(T)}{\alpha \int f d\mu} \quad \text{a.e.}$$

Proof. By Theorem 2.3 we have only to show that

$$\liminf_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(f)(x)} \geq \frac{h_\mu(T)}{\alpha \int f d\mu}.$$

For any C , if n is large, then by (3.2)

$$S_n(x) \leq C n^{\alpha+\epsilon}.$$

Hence for large k

$$\bar{R}_k(x) = S_{R_k(x)}(x) \leq C(R_k(x))^{\alpha+\epsilon}.$$

Taking logarithm

$$\log \bar{R}_k(x) \leq (\alpha + \epsilon) \log R_k(x) + \log C.$$

which implies

$$\frac{\log R_k(x)}{\log \bar{R}_k(x)} \geq \frac{1}{\alpha + \epsilon} \left(1 - \frac{\log C}{\log \bar{R}_k(x)} \right).$$

Hence we have

$$(3.3) \quad \liminf_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} \geq \frac{1}{\alpha}.$$

By applying Lemma 2.1 we obtain

$$\liminf_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(x)} \geq \frac{h_\mu(T)}{\alpha \mu(A)}.$$

The result follows replacing 1_A by Hopf's ergodic theorem with any $f \in L^1$ with $0 < \int f d\mu < \infty$. \square

4. MANNEVILLE POMEAU LIKE MAPS

In this section we consider a particular class of maps. We apply the results of the previous section to calculate recurrence time into balls. The maps we are going to consider are a generalization of the so called Manneville-Pomeau maps (Eq. 1.2) and are included in the class of maps studied by [27].

We say that a map $T : [0, 1] \rightarrow [0, 1]$ is a *Manneville-Pomeau map* (MP map) with exponent z if it satisfies the following conditions:

- (1) there is $c \in (0, 1)$ such that, if $I_0 = [0, c]$ and $I_1 = (c, 1]$, then $T|_{(0, c)}$ and $T|_{(c, 1)}$ extend to C^1 diffeomorphisms, $T(I_0) = [0, 1]$, $T(I_1) = (0, 1]$ and $T(0) = 0$;
- (2) there is $\lambda > 1$ such that $T' \geq \lambda$ on I_1 , whereas $T' > 1$ on $(0, c]$ and $T'(0) = 1$;
- (3) the map T has the following behaviour when $x \rightarrow 0^+$

$$T(x) = x + rx^z + o(x^z)$$

for some constant $r > 0$ and $z > 1$.

For example the map defined in equation 1.2 is in this family. In [27] it is proved that when $z \geq 2$ these maps have an infinite, absolutely continuous invariant measure μ with positive density and the entropy can be calculated as $h_\mu(T) = \int_{[0, 1]} \log(T') d\mu$.

These maps have DK sets where the first return map is mixing (see [1] page 103) and hence they satisfy the assumptions of the above section. If $z > 2$ (see e.g. [29]) we have a behavior of the return time sequence given by

$$a_n(T) = n^{1/(z-1)} s(n),$$

where $s(n)$ is a slowly varying function. Therefore setting $S_n(x) = \sum_{i \leq n} 1_{I_1}(T^i(x))$, we have by Theorem 3.2

$$\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(x)} = \frac{h_\mu(T)}{\mu(I_1)}(z-1).$$

Theorem 4.1. *Let (X, T, ξ) satisfy (1)-(3) and μ be the absolutely continuous invariant measure then*

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(x)}{-\log r} = z - 1$$

for almost all points x (we recall that $\tau_r(x)$ is the first return time of x in the ball $B(x, r)$).

In the proof of the above theorem we will use the following technical lemma on the behavior of real sequences.

Lemma 4.2. *Let r_n be a decreasing sequence such that $r_n \rightarrow 0$ and there is a $c > 0$ such that $r_{n+1} > cr_n$ eventually as n increases. Let $\tau_r : \mathbb{R} \rightarrow \mathbb{R}$ be decreasing. Then $\lim_{n \rightarrow \infty} \frac{\log \tau_{r_n}}{-\log r_n} = l \implies \lim_{r \rightarrow 0} \frac{\log \tau_r}{-\log r} = l$.*

Proof. If $r_n \geq r \geq r_{n+1} \geq cr_n$ then $\tau_{r_{n+1}} \geq \tau_r \geq \tau_{r_n}$, moreover $\log r_n \geq \log r \geq \log r_{n+1} \geq \log cr_n \geq \log c + \log r_n$ hence for n big enough

$$\frac{\log \tau_{r_{n+1}}}{-\log r_{n+1}} \geq \frac{\log \tau_r}{-\log r - \log c} \geq \frac{\log \tau_{r_n}}{-\log r_n - 2 \log c},$$

which taking the limits gives the statement.

□

Proof of Theorem 4.1. By theorem 3.2, for almost each x ,

$$\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n(x)} = \frac{h_\mu(T)}{\mu(I_1)}(z - 1).$$

Moreover, by the infinite measure version of the SMB theorem (see e.g. [27] or [29])

$$\frac{-\log(\mu(\xi_n(x)))}{S_n(x)} \rightarrow \frac{h_\mu(T)}{\mu(I_1)}.$$

Dividing the two equations we have

$$\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{-\log(\mu(\xi_n(x)))} = z - 1.$$

Now, since the measure is absolutely continuous with positive density (which is also bounded near each point $x \neq 0$) and since each element of the partition $\xi_n(x)$ is an interval, the diameter of such an element is equal to its measure up to a multiplicative constant (which depends on x) and then

$$\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{-\log(\text{diam}(\xi_n(x)))} = z - 1$$

for almost all x . Since obviously $\xi_n(x) \subset B(x, \text{diam}(\xi_n(x)))$, we have that

$$\tau_{\text{diam}(\xi_n(x))}(x) \leq R_n(x)$$

and then $\limsup_{n \rightarrow \infty} \frac{\tau_{\text{diam}(\xi_n(x))}(x)}{-\log(\text{diam}(\xi_n(x)))} \leq z - 1$. Now, we have that $\text{diam}(\xi_n(x))$ is a decreasing sequence, moreover by the SMB theorem for almost each x it holds $\text{diam}(\xi_n(x)) \sim 2^{-\text{Const.} * S_n}$. Then $\text{diam}(\xi_n(x)) \rightarrow 0$, moreover, since S_n increases at most by 1 at each step then when n is big there is a constant $c > 0$ such that $\text{diam}(\xi_{n+1}(x)) \geq c * \text{diam}(\xi_n(x))$. Now from 4.2 we have $\limsup_{r \rightarrow 0} \frac{\log \tau_r(x)}{-\log r} \leq z - 1$.

For the opposite inequality we consider the set

$$J_n = \{x : d(x, \partial \xi_n(x)) \geq \frac{1}{n^2} \text{diam}(\xi_n(x))\}$$

which is a set made of the union of ‘central’ parts of each interval $\xi_n(x)$. We have that $\lambda(J_n) \geq 1 - \frac{2}{n^2}$ (λ denotes the Lebesgue measure) and then by the Borel-Cantelli principle there is a full measure set J such that $x \in J$ implies that there is n s.t. $x \in J_k, \forall k \geq n$. Now if $x \in J_n$ then $\tau_{\frac{1}{n^2} \text{diam}(\xi_n(x))}(x) \geq R_n(x)$, hence

$$\liminf_{n \rightarrow \infty} \frac{\log(\tau_{\frac{1}{n^2} \text{diam}(\xi_n(x))}(x))}{-\log(\text{diam}(\xi_n(x)))} \geq z - 1.$$

By Theorem 2.2 we have $\liminf \frac{S_n}{\log(n)} = \infty$ almost everywhere and then, by the SMB theorem $\log(n) = o(-\log \text{diam}(\xi_n(x)))$ almost everywhere. Then we have

$$\lim_{n \rightarrow \infty} \frac{\log(\tau_{\frac{1}{n^2} \text{diam}(\xi_n(x))}(x))}{-\log \text{diam}(\xi_n(x))} = \lim_{n \rightarrow \infty} \frac{\log(\tau_{\frac{1}{n^2} \text{diam}(\xi_n(x))}(x))}{-\log(\frac{1}{n^2} \text{diam}(\xi_n(x)))} \geq z - 1,$$

then, applying Lemma 4.2 as before we have the statement. □

5. A COUNTEREXAMPLE

In this section we construct an example of a system having a “pathological” behavior, for which $\lim_{n \rightarrow \infty} \frac{\log S_n}{\log n}$ does not exist, and then also $\lim_{n \rightarrow \infty} \frac{\log R_n(x)}{S_n}$ does not exist (it oscillates, having a different \limsup and \liminf behavior). The system is constructed in an abstract way. We are not aware of any smooth, or “concrete” systems presenting such a behavior.

Recall from Section 2 that if $\liminf_n \frac{\log S_n}{\log n} = \alpha$, then $\limsup_n \frac{\log R_n}{S_n} = \frac{h_\mu(T)}{\alpha\mu(A)}$. To show that $\lim_n \frac{\log R_n}{S_n}$ does not exist when $\frac{\log S_n}{\log n}$ does not converges, we show the following lemma with the same assumptions in Section 2:

Lemma 5.1. *If $\limsup_{n \rightarrow \infty} \frac{\log S_n}{\log n} = \beta$ a.e., then we have*

$$\liminf_{n \rightarrow \infty} \frac{\log R_n}{S_n} = \frac{h_\mu(T)}{\beta\mu(A)} \quad \text{a.e.}$$

Proof. If $\limsup_n \frac{\log S_n}{\log n} = \beta$, then for any $\epsilon > 0$ $\lim_n \frac{S_n}{n^{\beta+\epsilon}} = 0$ a.e., which implies that

$$\liminf_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} \geq \frac{1}{\beta}.$$

The proof is similar to Theorem 2.3. for the upper bound, we suppose that there is $\lambda < 1$ such that

$$\liminf_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} \geq \frac{1}{\lambda^3\beta}.$$

Hence $(R_k(x))^{\lambda^2\beta} > \bar{R}_k(x)$ for all k large enough. Since $\bar{R}_k(x) = S_{R_k(x)}(x)$, if $S_n = \bar{R}_k$ and $n = R_k$ for large k , we have $S_n < n^{\lambda\beta}$. By the assumption we have infinitely many n 's such that $S_n > n^{\lambda\beta}$. Hence, there are infinitely many m 's such that

$$(5.1) \quad \bar{R}_k \notin (m^\lambda, m) \text{ for any } k.$$

By (2.4) if k is large, we have $\bar{R}_{k+1}^\lambda < \bar{R}_k$ which contradicts to (5.1). Therefore, we have

$$\liminf_{k \rightarrow \infty} \frac{\log R_k(x)}{\log \bar{R}_k(x)} = \frac{1}{\beta}.$$

Combined with Lemma 2.1, we have

$$\liminf_{n \rightarrow \infty} \frac{\log R_n}{S_n} = \frac{h_\mu(T)}{\beta\mu(A)} \quad \text{a.e.}$$

□

We will sketch the construction of an example (X, \mathcal{F}, μ, T) which is ergodic and measure preserving with the following properties.

1. There exists a subset X_0 of finite measure where $T|_{X_0}$ is Bernoulli (independent process) with respect to the partition $\{P_0, P_1\}$.
2. The partition $\mathcal{P} = \{P_0, P_1, P_2\}$ of X generates the σ -algebra where $X_0 = P_0 \cup P_1$ and $X_0^c = P_2$. That is, if x and y are different points of X , then there exists a time k such that $T^k x$ and $T^k y$ belong to different atoms of \mathcal{P} .
3. Let $0 < d < \frac{1}{2}$. For a.e. $x \in X$, $\liminf \frac{S_n(x)}{n^{1/2}} < 1$ and

$$\limsup \frac{S_n(x)}{n^{1/2+d}} > 1,$$

where $S_n(x) = \#\{0 \leq i \leq n-1 : T^i x \in X_0\}$.

Since it is ergodic, Property 3 says that the measure of X should be infinite. We start with $\mu(X_0) = 1$. We will go through the successive construction of X_i 's where $X_0 \subset X_1 \subset X_2 \subset \dots$. Each one of them has finite measure, but $X = \cup X_i$ has infinite invariant measure. Let S denote the independent process of 0's and 1's on $X_0 = P_0 \cup P_1$, where $\mu(P_0) = \mu(P_1) = \frac{1}{2}$. We will construct the space X and T by "exducing" over (X_0, S) . That is, we build X by constructing a tower over X_0 and we let the whole tower belong to the set P_2 .

We will briefly describe the first two steps using the names of 0's, 1's and 2's. For $x \in X_0$, we consider N_1 -long name of x , $x_{[0, N_1]} = x_0, \dots, x_{N_1-1}$ where $x_i = j$ if $T^i x \in P_j$, $j = 0$ or 1 . For each x , we construct

$$\tilde{x}_{[0, \tilde{N}_1]}^{(1)} = x_0 \dots x_{N_1-1} \underbrace{2 \dots \dots 2}_{\tilde{N}_1 - N_1}.$$

Let $(X_1, \mathcal{F}_1, \mu_1, T_1)$ denote the "exduced" map where X_1 is the union of all $\tilde{x}^{(1)}$'s. Clearly, we have $\sup_{0 \leq n < N_1-1} \frac{S_n(x)}{n^{1/2+d}} > 1$ for a.e. $x \in X_0$. By choosing $\tilde{N}_1 - N_1$ long enough, we also have $\inf_{0 \leq n < \tilde{N}_1-1} \frac{S-n(x)}{n^{1/2}} < 1$ for a.e. $x \in X_0$.

In the second step, for $\tilde{x}^{(1)} \in X_1$, we let $N_2 = N_1 \ell_2$ and

$$\tilde{x}_{[0, N_2]}^{(2)} = \overbrace{x_0 \dots x_{N_1-1} 2 \dots 2}^{N_1} \overbrace{x_{N_1} \dots x_{2N_1-1} 2 \dots 2}^{N_1} \dots \overbrace{x_{\ell_2 N_1-1} 2 \dots 2}^{N_1}$$

and

$$\tilde{x}_{[0, \tilde{N}_2]}^{(2)} = \tilde{x}_{[0, N_2]}^{(1)} \underbrace{2 \dots \dots \dots 2}_{\tilde{N}_2 - N_2}$$

Since $\mu_1(X_0) > 0$, for any given $0 < \delta < 1$, it is not difficult to see that if we make ℓ_2 big enough, then we have the set $\{x \in X_1 : \sup_{N_1 \leq n < N_2} \frac{S_n(x)}{n^{1/2+d}} > 1\}$ has measure bigger than $(1 - \delta^2)\mu_1(X_1)$. Also, we choose $\tilde{N}_2 - N_2$ large enough so that we have $\inf_{\tilde{N}_1 \leq n < \tilde{N}_2} \frac{S_n(x)}{n^{1/2}} < 1$ for a.e. $x \in X_1$. We repeat this process so that we have at the n -th stage,

$$(i) \{x \in X_i : \sup_{N_{n-1} \leq n < N_n} \frac{S_n(x)}{n^{1/2+d}} > 1\}$$

has measure bigger than $(1 - \delta^n)\mu_i(X_i)$, $i = 0, \dots, n-1$ and

$$(ii) \inf_{N_{n-1} \leq n < N_n} \frac{S_n(x)}{n^{1/2}} < 1 \text{ for a.e. } x \in X_{n-1}.$$

We note that this is possible because $\frac{\mu_n(X_0)}{\mu_n(X_n)} > 0$ for each n . Since N_i and \tilde{N}_i both go to infinity, the final system (X, \mathcal{F}, μ, T) clearly satisfies the properties by the Borel-Cantelli lemma.

Remark 5.2. Our examples of infinite invariant measure seem to show close relations between S_n and R_n . Clearly this is due to the fact that both MP like maps and the above example have positive entropy. However in the case of zero entropy, not much is known for finite or infinite invariant measures[19].

REFERENCES

- [1] J. Aaronson and M. Denker, *Upper bounds for ergodic sums of infinite measure preserving transformations*, Trans. Amer. Math. Soc. **319** (1990), no. 1, 101–138.

- [2] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997.
- [3] V. Afraimovich, J. R. Chazottes, B. Saussol, *Pointwise dimensions for Poincaré recurrences associated with maps and special flows*, Disc. Cont. Dyn. Syst. - A **9** (2003), 263–280
- [4] L. Barreira and B. Saussol, *Hausdorff dimension of measures via Poincaré recurrence*, Commun. Math. Phys. **219** (2001), no. 2, 443–463.
- [5] C. Bonanno and S. Galatolo, *Algorithmic information for interval maps with an indifferent fixed point and infinite invariant measure*. Chaos **14** (2004), no. 3, 756–762.
- [6] C. Bonanno, S. Isola and S. Galatolo, *Recurrence and algorithmic information*, Nonlinearity **17** (2004), no. 3, 1057–1074.
- [7] M. Boshernitzan, *Quantitative recurrence results*, Invent. Math. **113** (1993), no. 3, 617–631.
- [8] Carletti T. and Galatolo S., *Numerical Estimates of dimension by recurrence and waiting time (work in preparation)*.
- [9] G.H. Choe, *A universal law of logarithm of the recurrence time*, Nonlinearity **16** (2003), no. 3, 883–896.
- [10] G.H. Choe, *Computational Ergodic Theory*, Algorithms and Computation in Mathematics 13, Springer 2005.
- [11] G.H. Choe and D.H. Kim, *The first return time test of pseudorandom numbers*, J. Comput. Appl. Math. **143** (2002), no. 2, 263–274.
- [12] J.R. Chazottes, E. Ugalde, *Entropy estimation and fluctuations of hitting and recurrence times for Gibbsian sources*, to appear in Discr. & Cont. Dynam. Sys. B.
- [13] S. Galatolo, *Dimension via waiting time and recurrence*, Math. Res. Lett. **12**, no 3, May 2005, 377–386
- [14] T. Geisel and S. Thomae *Anomalous diffusion in intermittent chaotic systems*. Phys. Rev. Lett. **52** (1984), no. 22, 1936–1939.
- [15] T.C. Halsey and M.H. Jensen, *Hurricanes and butterflies*, Nature **428**, 11 Mar. (2004), 127–128.
- [16] N. Haydn, J. Luevano, G. Mantica, S. Vaienti, *Multifractal properties of return time statistics*, Phys. Rev. Lett. **88**, No. 22, 3 June 2003.
- [17] D.H. Kim and B.K. Seo, *The waiting time for irrational rotations*, Nonlinearity **16** (2003), no. 5, 1861–1868.
- [18] C. Kim and D.H. Kim, *On the law of logarithm of the recurrence time*. Discrete Contin. Dyn. Syst. **10** (2004), no. 3, 581–587.
- [19] D.H. Kim and K.K. Park, *The first return time properties of an irrational rotation*, print.
- [20] E. Klimko, L. Sucheston, *On convergence of information in spaces with infinite invariant measure*. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **10** 1968 226–235.
- [21] U. Krengel *Entropy of conservative transformations*. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **7** 1967 161–181.
- [22] P. Manneville, *Intermittency, self-similarity and $1/f$ spectrum in dissipative dynamical systems*, J. Physique **41** (1980), 1235–1243
- [23] D. Ornstein and B. Weiss, *Entropy and data compression schemes*, IEEE Trans. Inform. Theory **39** (1993), 78–83.
- [24] D. Ornstein and B. Weiss, *Entropy and recurrence rates for stationary random fields*, Special issue on Shannon theory: perspective, trends, and applications. IEEE Trans. Inform. Theory **48** (2002), no. 6, 1694–1697.
- [25] B. Saussol, *Recurrence rate in rapidly mixing dynamical systems*, preprint.
- [26] B. Saussol, S. Troubetzkoy and S. Vaienti, *Recurrence, dimensions and Lyapunov exponents*, J. Stat. Phys. **106** (2002), 623–634.
- [27] M. Thaler. *Transformations on $[0, 1]$ with infinite invariant measures*. Israel J. Math. **46** (1983), no. 1-2, 67–96.
- [28] R. Zweimüller, *Asymptotic orbit complexity of infinite measure preserving transformations*, preprint.
- [29] R. Zweimüller, *Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points*. Ergodic Theory Dynam. Systems **20** (2000), no. 5, 1519–1549.

UNIVERSITY OF PISA, PISA 56100, ITALY

E-mail address: `galatolo@dm.unipi.it`

KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 130-722, KOREA

E-mail address: `kimdh@kias.re.kr`

AJOU UNIVERSITY, SUWON 443-749, KOREA

E-mail address: `kkpark@ajou.ac.kr`